Exercise 1 (Block encodings). Verify that if \( cU = \sum_t |t\rangle \langle t| \otimes U^t \) and \( |f\rangle = \sum_t \sqrt{f(t)} |t\rangle \), then
\[
(I \otimes |f\rangle \langle f|) cU |f\rangle \psi) = |f\rangle \left( \sum_t f(t) U^t |\psi\rangle \right).
\]
- Let \( V \) be an operator such that \( V |0\rangle = |f\rangle \). Use \( V \) and \( cU \) to construct a block encoding
  \[
  \left[ \sum_t f(t) U^t \right].
  \]
  This shows that LCU effectively implements a block encoding of the target operator.
- Consider input Hamiltonian \( H \) with \( \|H\|_1 \leq 1 \). We can access \( H \) through a “row oracle” \( V \) such that
  \[
  V |0\rangle |i\rangle = \sum_j \sqrt{H_{ij}} |i\rangle |j\rangle + \sqrt{1 - \sum_j |H_{ij}| |i\rangle \langle i|},
  \]
  where \( \langle k| \perp \rangle = 0 \) for all \( k \). Use \( V \) and the SWAP operator (defined by \( \text{SWAP} |a\rangle |b\rangle = |b\rangle |a\rangle \)) to construct a block encoding \( U \) of \( H \). Verify that the encoding is also a reflection \( U^2 = I \).

Exercise 2 (Chebyshev polynomials, hard). Consider a block encoding
\[
U = \begin{bmatrix} H \\ -I \end{bmatrix} \quad \text{with} \quad U^2 = I.
\]
Prove that
\[
\left( \begin{bmatrix} I & -I \end{bmatrix} U \right)^k = \begin{bmatrix} T_k(H) \\ 0 \end{bmatrix}.
\]
Use induction: \( T_{k+2}(x) = 2xT_{k+1}(x) - T_k(x) \) with \( T_1(x) = x \) and \( T_0(H) = 1 \).

\[^{\text{1}}\text{The root } \sqrt{H_{ij}} \text{ is chosen such that } \sqrt{H_{ii}} \sqrt{H_{ij}} = H_{ij}.\]