Quantum algorithmic techniques (exercises)

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Exercise 1 (Block encodings).

- *Verify that if* $cU = \sum_t |t\rangle\langle t| \otimes U^t$ *and* $|f\rangle = \sum_t \sqrt{f(t)}\,|t\rangle$ *then*

$$(I \otimes |f\rangle\langle f|)\, cU |f\rangle |\psi\rangle = |f\rangle \left(\sum_t f(t) U^t |\psi\rangle \right).$$

- *Let* V *be an operator such that* $V|0\rangle = |f\rangle$*. Use* V *and* cU *to construct a block encoding*

$$\begin{bmatrix} \sum_t f(t)U^t & \cdot \\ \cdot & \cdot \end{bmatrix}.$$

  This shows that LCU effectively implements a block encoding of the target operator.

- *Consider input Hamiltonian* H *with* $\|H\|_1 \leq 1$*. We can access* H *through a "row oracle"* V *such that*[1]

$$V|0\rangle|i\rangle = \sum_j \sqrt{H_{ij}}\,|i\rangle|j\rangle + \sqrt{1 - \sum_j |H_{ij}|}\,|i\rangle|\perp\rangle,$$

  where $\langle k|\perp\rangle = 0$ *for all* k*. Use* V *and the* SWAP *operator (defined by* SWAP $|a\rangle|b\rangle = |b\rangle|a\rangle$*) to construct a block encoding* U *of* H*. Verify that the encoding is also a reflection (*$U^2 = I$*).*

Exercise 2 (Chebyshev polynomials, hard). *Consider a block encoding*

$$U = \begin{bmatrix} H & \cdot \\ \cdot & \cdot \end{bmatrix} \qquad \text{with} \qquad U^2 = I.$$

Prove that

$$\left(\begin{bmatrix} I & \\ & -I \end{bmatrix} U \right)^k = \begin{bmatrix} T_k(H) & \cdot \\ \cdot & \cdot \end{bmatrix}.$$

Use induction: $T_{k+2}(x) = 2xT_{k+1}(x) - T_k(x)$ *with* $T_1(x) = x$ *and* $T_0(H) = 1$*.*

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[1] The root $\sqrt{H_{ij}}$ is chosen such that $\sqrt{H_{ji}}^* \sqrt{H_{ij}} = H_{ij}$.