# Quantum Brascamp-Lieb Inequalities

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# Overview

geometric inequalities $\longleftrightarrow$ entropy inequalities

Brascamp-Lieb inequalities have wide range of applications and satisfy beautiful duality. We study a quantum formulation, motivated by the desire to identify new tools to proving entropy inequalities.

Plan for today:

1. Introduction
2. Quantum BL duality, applications and connections
3. Geometric quantum BL inequalities

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# Classical Brascamp-Lieb inequalities

For $B_k \colon \mathbb{R}^m \twoheadrightarrow \mathbb{R}^{m_k}$ linear, $q_k > 0$, $C > 0$, an inequality of the form

$$\int_{\mathbb{R}^m} \prod_{k=1}^{n} |f_k(B_k x)| \, dx \leqslant C \prod_{k=1}^{n} \|f_k\|_{1/q_k} \qquad \forall f_k$$

This generalizes many classical integral inequalities (Hölder, Young, ...)
Many proofs, applications, variations...

- Optimal $C$ can be computed by optimizing over Gaussian $f_k$. [Lieb]
- When is $C$ finite? Fully classified. [Bennett et al]
- How to compute $C$ efficiently? Still partly open! [Garg et al]

**Geometric case:** $B_k$ projections s.th. $\sum_{k=1}^{n} q_k B_k^* B_k = I_m$.

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# Duality and entropy

BL inequality is dual to 'subadditivity' inequality for differential entropy:

$$\sum_{k=1}^{n} q_k \, S(B_k X) \geqslant S(X) - \log C \qquad \forall \text{ RV } X \text{ on } \mathbb{R}^m$$

Apart from information theoretic interest, equivalence also enables new proof techniques (heat flow). [Carlen–Cordero-Erausquin]

The duality can be generalized to arbitrary channels and relative entropies. Framework includes hypercontractivity, strong data processing, etc. [Liu et al]

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# Result: Quantum Brascamp-Lieb Duality

Let $\mathcal{E}_k \colon L(\mathcal{H}) \to L(\mathcal{H}_k)$ positive & TP, $q_k > 0$, $\sigma$, $\sigma_k \succ 0$, $C > 0$.
Then the following are **equivalent**:

$$\sum_{k=1}^{n} q_k \, D(\mathcal{E}_k(\rho) \| \sigma_k) \leqslant D(\rho \| \sigma) + \log C \qquad \forall \text{ states } \rho$$

and

$$\operatorname{tr} e^{\log \sigma + \sum_{k=1}^{k} \mathcal{E}_k^*(\log \omega_k)} \leqslant C \prod_{k=1}^{n} \| e^{\log \omega_k + q_k \log \sigma_k} \|_{1/q_k} \quad \forall \, \omega_k \succ 0$$

- Proof via Legendre: $D(\rho \| \sigma) = \sup_{\omega \succ 0} \{ \operatorname{tr} \rho \log \omega - \log \operatorname{tr} e^{\log \omega + \log \sigma} \}$ [Petz]
- Not clear which side looks more intimidating. . .
- Useful choices: $\sigma_k = \mathcal{E}_k(\sigma)$ or $\sigma_k = I$, $\sigma = I$

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# Without side information

[Carlen-Lieb]

When specializing to $\sigma_k = I$, $\sigma = I$, recover equivalence between

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$$\operatorname{tr} e^{\sum_{k=1}^{k} \mathcal{E}_k^*(\log \omega_k)} \leqslant C \prod_{k=1}^{n} \|\omega_k\|_{1/q_k} \qquad \forall\, \omega_k \succ 0$$

For example, can prove uncertainty relations via trace inequalities, as pioneered by Frank-Lieb:

- Maassen-Uffink: $S(X) + S(Z) \geqslant S(\rho) + 1$ via Golden-Thompson
- Six-state [Coles et al]: $S(X) + S(Y) + S(Z) \geqslant S(\rho) + 2$ via Lieb 3-matrix

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# Applications and questions

- Can we prove new uncertainty relations involving multiple measurements (and even general quantum channels)? $N$-matrix GT?

- Strong data-processing inequalities fall into the framework:

$$D(\mathcal{E}(\rho)\|\mathcal{E}(\sigma)) \leqslant \eta D(\rho\|\sigma) \qquad \forall\rho$$

- Tensorization holds classically, but fails quantumly:

$$(\mathcal{E}, C) \;\&\; (\mathcal{E}', C') \quad \not\Rightarrow \quad (\mathcal{E}\otimes\mathcal{E}', C\cdot C')$$

  Examples include non-additivity of minimal output entropy. Useful?

- Computational complexity of testing validity of (families of) BL ineqs?
- Relation to works by Carlen-Maas?

# Back to geometry...

Recall the classical Brascamp-Lieb inequalities in the geometric case:

$$\sum_{k=1}^{n} q_k \, S(P_k X) \geqslant S(X) \qquad \forall \text{ RV } X \text{ on } \mathbb{R}^m$$

with $P_k$ projections onto subspaces $V_k \subseteq \mathbb{R}^m$ s.th. $\sum_{k=1}^n q_k P_k = I_m$.

How can we formulate a quantum version? For any subspace $V \subseteq \mathbb{R}^m$,

$$L^2(\mathbb{R}^m) = L^2(V \oplus V^\perp) = L^2(V) \otimes L^2(V^\perp)$$

hence can define reduced state $\rho_V$ for any state $\rho$ on $L^2(\mathbb{R}^m)$.

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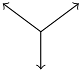
# Result: Geometric Quantum Brascamp-Lieb Inequality

**Theorem**

Let $P_k$ projections onto subspaces $V_k \subseteq \mathbb{R}^m$ s.th. $\sum_{k=1}^n q_k P_k = I_m$. Then, for all states $\rho$ on $L^2(\mathbb{R}^m)$ with finite first and second moments:

$$\sum_{k=1}^{n} q_k \, S(\rho_{V_k}) \geqslant S(\rho)$$

- For coordinate subspaces recover quantum Shearer inequality. [Carlen-Lieb]
- But already nontrivial for "Mercedes star" configuration in $\mathbb{R}^2$:



- Also holds conditioned on side information. [Ligthard]
- Can generate more ineqs. via Gaussian unitaries: $\mathrm{Sp}_{2m} \curvearrowright L^2(\mathbb{R}^m)$...

# Sketch of proof

$$\sum_{k=1}^{n} q_k \, S(\rho_{V_k}) \geqslant S(\rho)$$

Implement classical proof strategy of Carlen–Cordero-Erausquin using quantum heat flow of König-Smith: cf. [De Palma–Trevisan]

$$\frac{d}{dt}\rho = -\sum_{j=1}^{m} [Q_j, [Q_j, \rho]] + [P_j, [P_j, \rho]]$$

Asymptotic scaling of entropy: $S(\rho_V(t)) \sim \dim V \log t$

- Inequality holds at $t = \infty$ if $\sum_k q_k \dim V_k \geqslant m$.

Quantum de Bruijn identity: $\frac{d}{dt} S(\rho) = J(\rho)$, a Fisher information.

- Can prove reverse inequality for Fisher information if $\sum_k q_k P_k \leqslant I_m$:

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## Gaussian BL beyond the geometric case

[De Palma–Trevisan]

There is a natural action of $\mathrm{Sp}_{2m}$ on $L^2(\mathbb{R}^m)$ by Gaussian unitaries. Any symplectic matrix $B \in \mathbb{R}^{2m' \times 2m}$ determines subsystem of $m'$ modes, so we can define reduced state $\rho_B$ on $L^2(\mathbb{R}^{m'})$ for any state $\rho$ on $L^2(\mathbb{R}^m)$.

This notion generalizes the reduced state $\rho_V$ for subspaces $V \subseteq \mathbb{R}^m$ and leads naturally to the following class of Gaussian quantum BL inequalities:

$$\sum_{k=1}^{n} q_k S(\rho_{B_k}) \geqslant S(\rho) + c$$

where the $B_k \in \mathbb{R}^{2m_k \times 2m}$ symplectic matrices. When does it hold?

**Recent result (De Palma–Trevisan):** Assuming $\sum_{k=1}^n q_k m_k = m$, inequality holds for all quantum states iff holds for all probability densities!

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- Also holds conditioned on side information.
- Can also include "classical" outputs (= quadrature measurements)
- Proof again based on quantum heat flow strategy!

# Outlook

$$\boxed{\text{trace inequalities}} \quad \underset{\textit{duality}}{\overset{BL}{\longleftrightarrow}} \quad \boxed{\text{entropy inequalities}}$$

Duality between quantum relative entropy inequalities and trace inequalities. Unifying framework to tackle information theoretic questions. New family of geometric quantum Brascamp-Lieb inequalities.

Many exciting directions:

- Uncertainty relations from *n*-matrix GT?
- Sufficient conditions for tensorization?
- Applications of new trace inequalities?
- Other applications of quantum heat flow?
- ...

*Thank you for your attention!*