Multipartite entanglement in toy models of holography

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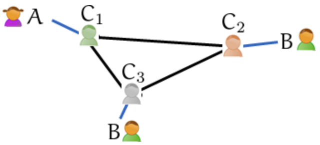
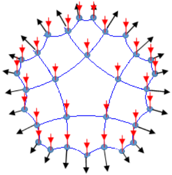
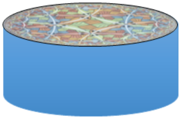
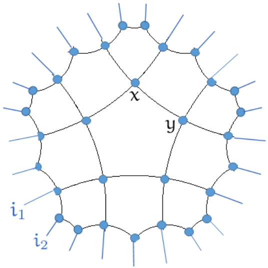






The random tensor network model

Given a graph $G = (V, E)$ and bond dimension 2^N, we consider



$$|\Psi\rangle = \left(\bigotimes_{\langle xy \rangle \in E} \langle xy| \right) \left(\bigotimes_{x \in V} |V_x\rangle \right)$$

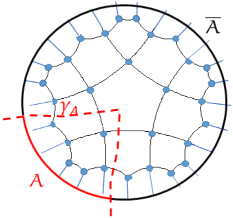
- $|V_x\rangle$ random tensors
- $|xy\rangle = \left(|00\rangle + |11\rangle\right)^{\otimes N}$ EPR pairs

We are interested in the behavior for large N.

Related work: Swingle (MERA with expanders), Collins *et al* (random MPS), Hastings (random MERA), . . . , HaPPY (perfect tensors)

Bipartite entanglement

Fundamental bound: $S(A) \leq N \min|\gamma_A|$



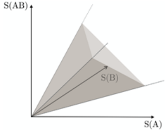
Result

In random tensor networks: $S(A) \simeq N \min|\gamma_A|$ with high probability

- Toy models of holography that reproduce Ryu-Takayanagi formula, including bulk corrections, satisfy the desired quantum error correction properties, etc. $\rightsquigarrow$ Xiaoliang's talk

Holographic entropy inequalities

The Ryu-Takayanagi formula has interesting structural properties.



$$S(A) = c \min|\gamma_A|$$

Can be studied systematically via entropy cone formalism:

- many nonstandard entropy inequalities – but finite number for any number of subsystems (with Bao, Ooguri, Stoica, Sully)
- can constrain QIT protocols (Czech *et al*) but also holographic CFTs
- ex.: monogamy of mutual information (Hayden, Headrick, Maloney)

$$I(A:B) + I(A:C) \leq I(A:BC)$$

is unique additional inequality for three subsystems.[1] But correlations are not in general monogamous – not valid for Shannon, vN entropy.

[1] $I(A:B) = S(A) + S(B) - S(AB)$

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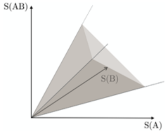

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Does the mutual information in these states measure entanglement?

Interlude: Some entanglement theory

For bipartite pure states, the EPR pair is basic unit of entanglement:

$$\psi_{AB}^{\otimes n} \overset{LOCC}{\longleftrightarrow} EPR^{\otimes m} \quad \textit{at rate} \quad \frac{m}{n} = S(A)_\psi$$

LOCC: local operations and classical communication.

However, in our case $I(A:B)$ will refer to a mixed state, with some tripartite purification $|\psi_{ABC}\rangle$.

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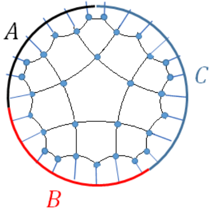

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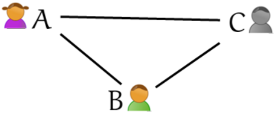
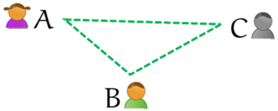
Interlude: Some multipartite entanglement theory

Let us compare three EPR pairs vs. a GHZ state:



$|\beta\rangle = \sum_{i,j,k} |ij\rangle\,|jk\rangle\,|ik\rangle$



$|\mathrm{GHZ}\rangle = \sum_i |iii\rangle$

"Clearly", the states cannot be inter-converted by local unitaries. However:

- LOCC: $\beta^{\otimes 2} \longrightarrow \mathrm{GHZ}^{\otimes 3}$, $\mathrm{GHZ}^{\otimes 3} \longrightarrow \beta$.
- SLOCC: $\beta \longrightarrow \mathrm{GHZ}^{\otimes 2}$, $\mathrm{GHZ}^{\otimes \leq 2.3728639} \longrightarrow \beta$.

The operational perspective matters – what transformations and tasks do we care about? (shameless ad: upcoming review with Gross & Eisert)

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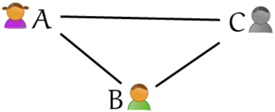

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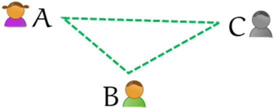

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Stabilizer states $D = 2^n$

From now on: we use random stabilizer states as the vertex tensors $|V_x\rangle$. Then the tensor network state $|\Psi\rangle$ is also a stabilizer state.

Stabilizer states: Eigenvector of maximal subset of Pauli operators.

Ex: $|\mathrm{GHZ}\rangle = |000\rangle + |111\rangle$ is stabilized by $X_1X_2X_3$, Z_1Z_2, Z_2Z_3.

- Useful for codes (incl. HaPPY ones), efficient random constructions, ...
- Reason: 3-design (for qubits only)
- Tripartite entanglement structure is simple:

$$I(A:B) = 2c + g$$

where g is the number of GHZ states.

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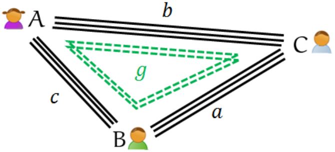

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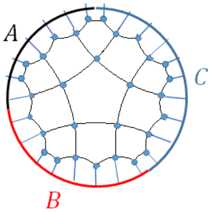

Consequence

Can distill $\simeq \frac{1}{2} I(A:B)$ EPR pairs by local unitaries.

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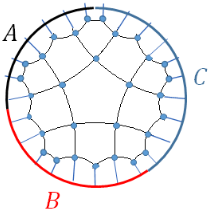

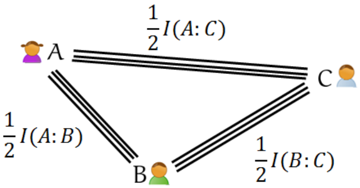

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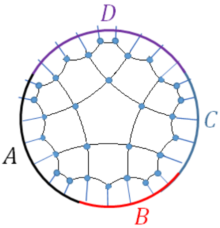

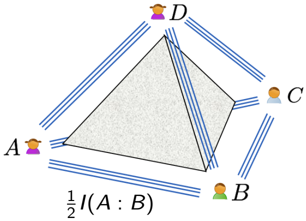

After distilling bipartite EPR pairs, we obtain residual state:

$$S(A), \ldots, S(D) \simeq -\tfrac{1}{2} I_3, \quad S(AB), \ldots, S(CD) \simeq -I_3$$

with the tripartite information $I_3 = I(A:B) + I(A:C) - I(A:BC)$:

- residual state has entropies of perfect tensor
- I_3 is invariant under distillation: can estimate via Ryu-Takayanagi
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- $I_3 < 0$ diagnoses four-partite entanglement

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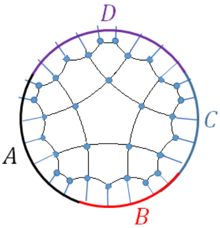

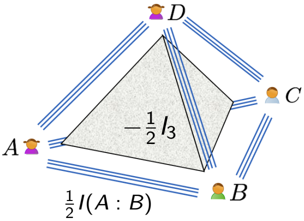

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Proof ingredient I: Spin models

Result (Bipartite entanglement)

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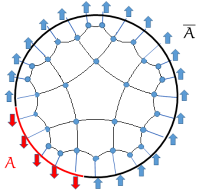
Sketch of proof: Lower-bound Rényi-2 entropy $S_2(A) = -\log \operatorname{tr} \rho_A^2$.

Using replica trick and second moments:

$$\mathbb{E}[\operatorname{tr} \rho_A^2] \propto Z_A = \sum_{\{s_x\}} e^{-N \sum_{\langle xy \rangle} (1 - s_x s_y)/2}$$



Ferromagnetic Ising model at $T = 1/N$ with mixed boundary conditions.

- $S_2(A)$ is related to free energy $F = -\log Z_A$
- large N/low T: dominated by energy of minimal domain wall

Useful general technique. More precise estimates possible in terms of geometry of graph. Exact limiting eigenvalue distribution available.

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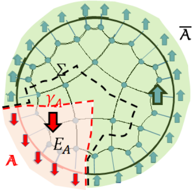

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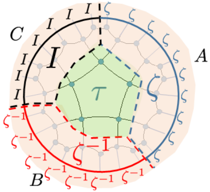

- ferromagnetic spin model with variables $\pi_x \in S_3$, cyclic boundary conditions
- minimal energy configuration displayed on the right

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- three-fold degenerate for every residual region (independent of large N)

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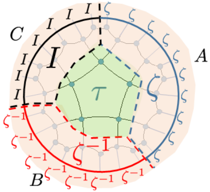

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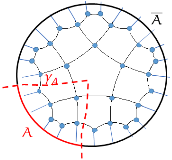
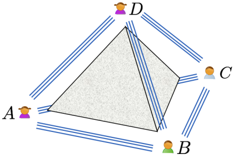
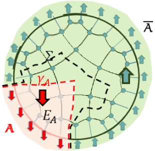
Summary and Outlook

arXiv:1608.02595







Random tensor networks:

- Bipartite & multipartite entanglement properties dictated by geometry
- Techniques: spin models for random tensor averages, stabilizer states

Beyond toy models:

- Design diagnostics that are both meaningful and computable
- Protocols! What is the holographic dual of an entanglement transformation?
- Emergence of the boundary state from entanglement distillation

Thank you for your attention!