Quantum marginal problem, tensor scaling, and invariant theory

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QuSoft
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Outline and philosophy

Quantum marginal problem $\longleftrightarrow$ Null cone problem

(Geometry) (Invariant theory)

Two dual problems and an algorithm that solves them: Tensor scaling

Philosophy:

- *An old duality,*[†] *recognized as such, leads to efficient new algorithms.*
- *'Computational invariant theory without computing invariants.'*

[†] Known since the 80s in algebraic geometry!

Warm-up: Horn's problem

Let $\alpha_1 \geq \ldots \geq \alpha_n \geq 0$, $\beta_1 \geq \ldots \geq \beta_n \geq 0$, $\gamma_1 \geq \ldots \geq \gamma_n \geq 0$ be integers.

Horn's problem (Geometry): When $\exists$ Hermitian $n \times n$ matrices A, B, C with spectrum α, β, γ such that $A + B = C$?

Horn proposed linear inequalities on α, β, γ.

Saturation property (Invariant theory): $\exists A, B, C$ iff *Littlewood-Richardson coefficient* $c^{\gamma}_{\alpha,\beta} > 0$ (Knutson-Tao)

- Horn inequalities sufficient
- lead to *only known* poly-time algorithm (Mulmuley)

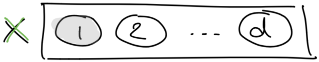
Today's talk is about a generalization to tensors!

Geometry: Quantum states and marginals

Quantum state of d particles is described by unit vector

$$X \in V = (\mathbb{C}^n)^{\otimes d} = \mathbb{C}^n \otimes \ldots \otimes \mathbb{C}^n$$

$$\rightsquigarrow [X] = |X\rangle \langle X| \in \mathbb{P}(V)$$



Quantum marginals: $n \times n$-matrices $\rho_1^X, \ldots, \rho_d^X$ that describe state of individual particles:

$$\operatorname{tr}[\rho_1^X A_1] = \langle (A_1 \otimes I \otimes \ldots \otimes I) X, X \rangle \quad \forall A_1$$

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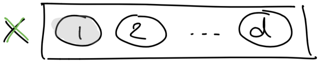

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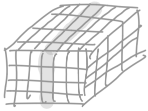

- $\rho_1^X = M_1 M_1^*$ if we ‘flatten’ X to $n \times n^{d-1}$ matrix M_1
- eigenvalues form probability distributions

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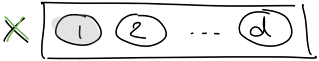

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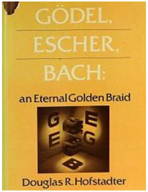
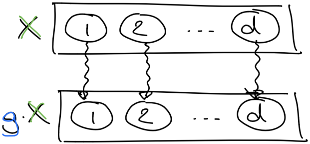


Quantum marginal problem: Which $(\rho_1^X, \ldots, \rho_d^X)$ can arise?

A natural group action

$X \in V = (\mathbb{C}^n)^{\otimes d}$

$G = \mathrm{SL}(n)^d$ acts on $V = (\mathbb{C}^n)^{\otimes d}$ by $g_1 \otimes \ldots \otimes g_d$



Group orbit = states that we can obtain by *local operations and classical communication*.

Which $(\rho_1^Y, \ldots, \rho_d^Y)$ can arise in orbit (closure)?

Problem 1

Given X, $\exists [Y] \in \overline{G \cdot [X]}$ such that $\rho_1^Y = \ldots = \rho_d^Y = \frac{I}{n}$?

- Quantum version of stochastic tensor
- Every particle is maximally entangled with rest

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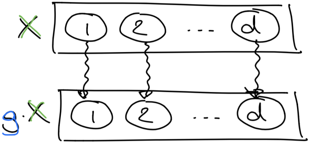

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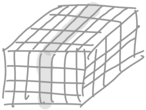
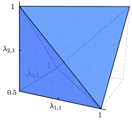


Quantum marginal polytopes

More generally, study

$$\Delta(X) = \{(p_1, \ldots, p_d) : p_i = \mathrm{spec}(\rho_i^Y),\ [Y] \in \overline{G \cdot [X]}\} \subseteq \mathbb{R}^{dn}$$

- Convex (moment) polytopes (Kirwan/Mumford)
- Inequalities ‘known’, but ‘intractable’ for $n > 4$ (Berenstein-Sjamaar, Klyachko, Ressayre, Vergne-W.)
- Can replace $\mathcal{X} = \overline{G \cdot [X]}$ by other $\mathcal{X} \subseteq \mathbb{P}(V)$...



Result (informal)

An efficient algorithm for deciding if a given point is in $\Delta(X)$.

Polytopes are of fundamental interest in quantum physics: related to entanglement distillation, monogamy of entanglement, Pauli principle, ... (but also: next talk)

Invariant theory

$G = \mathrm{SL}(n)^d$ acts on $V = (\mathbb{C}^n)^{\otimes d}$, so also on polynomials $\mathbb{C}[V]$

Problem 2

Given X, $\exists P \in \mathbb{C}[V]^G$ such that $P(X) \neq P(0)$?

- If no: $X \in$ null cone (geometric invariant theory)
- Even interesting for X generic!
- Equivalent: $\overline{G \cdot X} \not\ni 0$
- Algorithms: generators of $\mathbb{C}[V]^G$ or Hilbert-Mumford criterion & Gröbner bases → 'intractable' beyond small n.

The Kempf-Ness theorem

$G = \mathrm{SL}(n)^d$

Problem 1

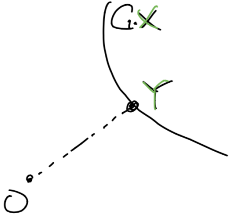
Given X, $\exists [Y] \in \overline{G \cdot [X]}$ s.th. $\rho_1^Y = \ldots = \rho_d^Y = \frac{I}{n}$?

Problem 2

Given X, is $\overline{G \cdot X} \not\ni 0$?

The two problems are equivalent! (Kempf-Ness)

($\Leftarrow$)



For all $A = (A_1, \ldots, A_d)$ Hermitian & traceless:

$$0 = \frac{1}{2}\partial_t \|e^{At} \cdot Y\|^2 = \sum_{i=1}^{d} \mathrm{tr}[\rho_i^Y A_i] \Rightarrow \rho_i^Y = \frac{I}{n} \; \forall i$$

($\Rightarrow$) Convexity properties...

Similar equivalence for entire polytope.

Towards an algorithm

Interpret Kempf-Ness theorem as duality between two optimization problems *(a noncommutative version of Farkas' lemma)*!

$$\boxed{\inf_{g \in G} ds(g \cdot X) = 0} \quad \longleftrightarrow \quad \boxed{\inf_{g \in G} \|g \cdot X\| > 0}$$

$$ds(Y) := \sum_{i=1}^{d} \|\rho_i^Y - \tfrac{I}{n}\|^2$$

Idea: Construct sequence of tensors $Y^{(0)} = X,\ Y^{(1)}, \ldots \in G \cdot X$ such that

$$\|Y^{(0)}\| > \|Y^{(1)}\| > \cdots > \|Y^{(t)}\| \to 0 \quad \text{unless} \quad ds(Y^{(t)}) \to 0$$

- *either* proves primal *or* disproves dual hypothesis
- elementary tensor scaling step:

  $Y^{(t+1)} \leftarrow (n\rho_i^{Y^{(t)}})^{-1/2} \cdot Y^{(t)}$

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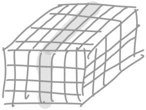

Our result

Theorem

A poly($\frac{1}{\varepsilon}$, input size)-time algorithm

- Input: $X \in V$ and $\varepsilon > 0$
- Output: $g \in G$ s.th. $ds(g \cdot X) < \varepsilon$, *or* certificate that X in null cone.

- If ε chosen suitably small: $ds(g \cdot X) < \varepsilon$ implies that $\inf ds = 0$
- First exp-time algorithms for quantum marginal problem, asymptotic support of Kronecker coefficients, convex optimization over moment polytopes ($\rightsquigarrow$ Jeroen's talk), ...
- Easily adapted to structured tensors (e.g., matrix product states)

Analysis via quantitative version of AM/GM inequality and new a priori bounds on the complexity of invariants and highest weight vectors.

Summary and outlook

arXiv:1711.08039

Quantum marginal problem	$\overset{\text{duality}}{\longleftrightarrow}$	Null cone problem
when are $\rho_1, \dots, \rho_d$ compatible?		vanishing of invariants

Tensor scaling: Effective numerical (but rigorous) algorithm.

Computational invariant theory without computing invariants!

Many open questions:

- Poly-time algorithm? Quantum algorithm? — poly($\frac{1}{\varepsilon}$) *vs* poly($\log \frac{1}{\varepsilon}$)
- Other groups and representations? — Sym, $\bigwedge$, ...
- $\mathbb{C} \rightsquigarrow \mathbb{F}$?
- *What are the 'tractable' problems in invariant theory?*

Thank you for your attention!

The tensor scaling algorithm

Input: $X \in V$ rational, $\varepsilon > 0$

- If any ρ_i^X is singular: Null cone ↯
- Set $Y^{(0)} := X$.
- For $t = 0, 1, \ldots, T$:
  - If $ds(T^{(t)}) < \varepsilon$: Success ☺
  - Choose i such that $\|\rho_i^{Y^{(t)}} - \frac{I}{n}\| > \frac{\varepsilon}{\sqrt{d}}$ and apply tensor scaling step:

$$Y^{(t+1)} \leftarrow (n\rho_i^{Y^{(t)}})^{-1/2} \cdot Y^{(t)}$$

- Null cone ↯

Other target spectra: Adjust tensor scaling step (in particular, use Cholesky square root) and randomize initial point.

A general equivalence $\mathcal{X} \subseteq \mathbb{P}(V)$

All points in $\Delta(\mathcal{X})$ can be described via invariant theory:

$$V_\lambda \subseteq \mathbb{C}[\mathcal{X}]_{(k)} \quad \Rightarrow \quad \frac{\lambda}{k} \in \Delta(\mathcal{X})$$

(λ highest weight, k degree)

- Can also study multiplicities $g(\lambda, k) := \# V_\lambda \subseteq \mathbb{C}[\mathcal{X}]_{(k)}$.
- This leads to interesting computational problems:

$g =?$	$g > 0?$	$\exists s > 0 : g(s\lambda, sk) > 0?$
(#-hard)	(NP-hard)	(our problem!)

Completely unlike Horn's problem: *Knutson-Tao saturation property does not hold, and hence we can hope for efficient algorithms!*